Decentralizing the Growth Model

I. The Representative Agent Model

In this model (which we will sometimes refer to as the RA model), the representative agent solves

$$\max_{C_t,K_t,I_t,L_t;\, t=0,\dots,\infty} E\left[\sum_{t=0}^{\infty}\beta^t U(C_t,L_t)\right] \tag{1}$$

subject to (Lagrange multipliers listed at the left)

$$\mu: \qquad g(C_t, I_t, K_{t-1}, L_t, A_t) = 0 \tag{2}$$

or

$$C_t + I_t = A_t f(K_{t-1}, L_t) \tag{3}$$

or

$$g(C_t, I_t) = A_t f(K_{t-1}, L_t) \tag{4}$$

or

$$C_t + I_t g\left(\frac{I_t}{K_{t-1}}\right) = f(K_{t-1}, L_t) \tag{5}$$

and

$$\nu: \qquad K_t - K_{t-1} = I_t - \psi K_{t-1}\,. \tag{6}$$

We also require $C_t, K_t, L_t \geq 0$, all t, though commonly the forms of U and g (in the general form (2)) are chosen so that these constraints are never binding. It may be reasonable to require also that $I_t \geq 0$. This constraint can be expected not to bind under some reasonable choices of g in (4), but for the other forms of the technology constraint it is likely to bind occasionally.

II. Decentralized Model: Consumer

Here the objective function is (1) as in the RA model, but the constraint has the consumer purchasing consumption goods from wages and asset returns, with no direct recognition of the technology constraint. The consumer's constraint, in the version of decentralization we will focus on, is

$$\lambda: \qquad C_t + B_t + Q_t S_t = w_t L_t + (1 + r_{t-1})B_{t-1} + (Q_t + \delta_t)S_{t-1} + \pi_t\,. \tag{7}$$

Here π is profit distributions from the firm, which is owned by the consumer/worker. There are two traded assets, a "share" and a bond. The share has dividends per share

δ that may or may not be the same as π. We have set up the model so that each individual is endowed with ownership of the typical firm (or a unit share in it). If the traded share is a share in the typical firm, then the individual has the option of selling off part of his endowment.

Note that other choices are possible here. We are giving the "firm" control of all technological decisions, both employment and investment. Another setup that is used sometimes is to have consumers purchasing capital goods directly and renting them to the firm. Then it is consumers who have to compare costs of capital goods to their expected future productivity. As we have this model set up, though, it is the firms that make this kind of calculation.

We do not restrict bond and equity holdings to be positive—firms and consumers may borrow. However, this means that we must put in place some kind of constraint that prevents firms and consumers from borrowing arbitrarily large amounts, paying off old debts always by borrowing more. The most natural way to motivate such a no-Ponzi constraint is to appeal to the transversality condition of the lender. We will therefore wait to write down the no-Ponzi constraints until after we have displayed transversality conditions.

III. Decentralized Model: Firm

We suppose that the firm has a utility for its profit distributions, given by $\phi(\pi_t)$, with $\phi' > 0$ and $\phi'' < 0$, and that it maximizes a discounted sum of these utilities:

$$\max_{\{C_t, K_t, L_t, B_t, S_t, I_t;\, t=0,\dots,\infty\}} E \sum_{t=1}^{\infty} \beta^t \phi(\pi_t)\ . \tag{8}$$

Its constraints are (2) (or one of its specialized forms), (6), and

$$\zeta: \qquad \pi_t = C_t - w_t L_t + B_t - (1 + r_{t-1})B_{t-1} + Q_t S_t - (Q_t + \delta_t)S_{t-1}\ . \tag{9}$$

Often (9) is solved for C_t, the result is substituted into (2), and the firm is treated as having a single constraint in which C does not appear.

IV. Consumer FOC's

The Euler equations are

$$\partial C: \qquad D_C U_t = \lambda_t \tag{10}$$

$$\partial L: \qquad D_L U_t = -\lambda_t W_t \tag{11}$$

$$\partial B: \qquad \lambda_t = \beta(1 + r_t) E_t[\lambda_{t+1}] \tag{12}$$

$$\partial S: \qquad Q_t \lambda_t = \beta E_t[\lambda_{t+1}(Q_{t+1} + \delta_{t+1})] \tag{13}$$

Here as elsewhere in these notes, we are using the shorthand notation that if the typical arguments for a function f at t are X_t, $f_t = f(X_t)$ Also we are using $D_C U$ as notation for $\partial U / \partial C$.

The transversality condition for the consumer[1] is

$$\limsup_{t\to\infty} \beta^t E[-\lambda_t dW_t] \le 0 \,, \tag{14}$$

where $W_t = B_t + Q_t S_t$ is the consumer's wealth and dW_t represents any feasible (feasible for the consumer, not necessarily for the whole economy) deviation of wealth from its value in the candidate equilibrium. Observe that, because λ will be positive in equilibrium, transversality will not be satisfied along any finite time path for W if unlimited borrowing is allowed. With unlimited borrowing, it would be feasible to make dW negative and let it grow more negative at a rate such that (31) would be violated.

V. Firm FOC's

The firm's constraints are (5), (6), and (9). The Lagrange multiplier for each constraint is given at the left of the constraint as displayed above. We could have used any of (2) through (4) instead of (5). We choose this version of the technology constraint because it is probably the most common in the literature, and the main point of what we do here does not depend on which version we choose. The Euler equations are then

$$\partial\pi: \qquad \zeta_t = \phi'_t \tag{15}$$

$$\partial C: \qquad \zeta_t = \mu_t \tag{16}$$

$$\partial L: \qquad \mu_t A_t D_L f_t = \zeta_t W_t \tag{17}$$

$$\partial I: \qquad \mu_t \cdot \left(g_t + \frac{I_t}{K_{t-1}} g'_t \right) = \nu_t \tag{18}$$

$$\partial K: \qquad \nu_t - \beta E_t[\nu_{t+1}\cdot(1-\psi)] = \beta E_t \left[\mu_{t+1} \left(A_{t+1} D_K f_{t+1} + \frac{I_{t+1}^2}{K_t^2} g'_{t+1} \right) \right] \tag{19}$$

$$\partial B: \qquad \zeta_t = \beta(1+r_t)E_t[\zeta_{t+1}] \tag{20}$$

$$\partial S: \qquad \zeta_t Q_t = \beta E_t[\zeta_{t+1}(Q_{t+1}+\delta_{t+1})] \,. \tag{21}$$

It is helpful in interpreting the FOC's with respect to K and I to rearrange them a bit. Notice that the left-hand side of the technology constraint in (5) defines a transformation curve between C and I, whose slope can be thought of as determining the shadow price of capital goods in terms of consumption goods. The implicit shadow price is easily seen to be just the term in parenthesis in (18), and it is convenient to define special notation for this shadow price:

$$P_K(t) = g_t + \frac{I_t}{K_{t-1}} g'_t \,. \tag{22}$$

[1]We need to use the more general form of the TVC here because wealth is not constrained to be positive.

The functional form for g is usually chosen so that $g(0) = 1$, $0 < g'(0) < \infty$, $g_t'' \geq 0$.[2] That makes P_K one when investment is zero. Now we can rewrite (19) (using (18) and dividing through by μ_t) as

$$P_K(t) = \beta E_t \left[\frac{\mu_{t+1}}{\mu_t} \left(P_K(t+1) \cdot (1 - \psi) + A_{t+1} D_K f_{t+1} + \frac{I_{t+1}^2}{K_t^2} g'_{t+1} \right) \right] \tag{23}$$

This is an asset valuation equation much like (21) and (20). From these latter two equations we see that the role of stochastic discount factor in the firm's asset valuations is being played by $\beta\zeta_{t+1}/\zeta_t$, which because of (16) is the same as $\beta\mu_{t+1}/\mu_t$. So (23) can be read as saying that the current price of a capital good is the discounted present value of its market value next period, adjusted for depreciation, plus a dividend-like term arising from the marginal product of capital, both directly in the conventional production function f and indirectly through its ability to reduce capital adjustment costs generated by g.

The firm's transversality condition is

$$\limsup_{t \to \infty} \beta^t E[-\mu_t dV_t] \leq 0 \,, \tag{24}$$

where $V_t = P_K(t)K_t - B_t - Q_t S_t$ is the value of the firm. Note that we are using the convention that when B or S is positive, it represents borrowing by the firm, asset accumulation by the consumer. (In deriving (24) we use the Euler equations to eliminate all the Lagrange multipliers other than μ.)

One natural way to introduce no-Ponzi finance constraints is to treat the firm's transversality condition, with the time path of K seen by the consumer as exogenously determined, as the consumer's no-Ponzi constraint, while the consumer's transversality condition (14) is treated as the the firm no-Ponzi constraint. This is not as obvious as it seems, because in this competitive model individual firms should be thought of as each dealing with many consumers and vice versa. But it is true that for any one consumer to take on debt at a rate of β^{-t} would require at least one firm to buy bonds (i.e. lend) at that rate. So setting the no-Ponzi conditions this way makes some sense. However a constraint like this, because it affects only limiting behavior as $t \to \infty$, could not in practice be enforced. So requiring that for both firms and consumers wealth (or in a growth model, the ratio of wealth to $P_K K$, say) remain above some fixed (negative) lower bound would be a more realistic specification.

VI. Matching centralized and decentralized model solutions

For the representative agent, the Euler equations with respect to C and L are

$$\partial C: \qquad D_C U_t = \mu_t^* \tag{25}$$

$$\partial L: \qquad D_L U_t = -\mu_t^* A_t D_L f_t \,. \tag{26}$$

[2]E.g., it is common to assume $g(x) = 1 + x$. Convexity of the set of attainable C, I pairs does not actually require $g'' \geq 0$, though this condition does imply convexity.

Though the constraints (5) and (6) occur in the same form in both problems, we do have to recognize that their Lagrange multipliers, which we have called μ and ν, are not necessarily the same for both models. We will preserve the original μ, ν for the decentralized model and use, as in the equations just above, μ^*, ν^* for the RA case. The first order conditions with respect to K and I are then in exactly the form of the corresponding equations (19) and (18) from the firm's optimization problem, with $*$'d Lagrange multipliers.

The representative agent's transversality condition[3] is

$$\lim_{t\to\infty} \beta^t E[\mu_t^* P_K(t) K_t] = 0 \,. \tag{27}$$

We want to check whether a solution to the RA problem (which in this one-agent model is also the social planner's solution) corresponds to an equilibrium of the decentralized model. This amounts to determining whether the additional variables (S, B, w, Q and r) and Lagrange multipliers (μ, ν, λ and ζ) that appear in the decentralized model can be chosen in such a way that the constraints and FOC's of the decentralized model are satisfied.

From (25) and (10), we conclude that we will have to have $\lambda = \mu^*$. Then from (11), and (17), and (16) we have

$$D_L U_t = -\lambda_t W_t = -\lambda_t \frac{\mu_t}{\zeta_t} A_t D_L f_t = -\lambda_t A_t D_L f_t \,. \tag{28}$$

With the $\lambda = \mu^*$ condition we have already deduced, this equation implies (17). So we have verified that we can satisfy (10), (11), (16) and (17) by setting $\lambda = \mu^*$ and $\zeta = \mu$. Because we know that the C process must be that of the RA solution, we can solve for r from (12), as

$$1 + r_t = \beta^{-1} \frac{D_C U_t}{E_t[D_C U_{t+1}]} \tag{29}$$

and for Q_t from (13), solving forward to obtain

$$Q_t = E_t\left[\sum_{t=0}^{\infty} \beta^s \Phi(t; t+s)\delta_{t+s}\right], \tag{30}$$

where we are assuming

$$\beta^s E_t[\Phi(t; t+s) Q_{t+s}] \underset{s\to\infty}{\longrightarrow} 0 \tag{31}$$

and using the notation

$$\Phi(t; t+s) = \frac{\lambda_{t+s}}{\lambda_t} = \frac{D_C U_{t+s}}{D_C U_t} \,. \tag{32}$$

[3]Here we can use the simplified form of the condition because we assume $\mu^* = \phi'$ and K are both always positive.

Since we are here just checking that we can find one decentralized solution that supports the RA allocation, it need not worry us at this point that the solution for Q in (30) might not be unique if (31) were violated.

The version of the capital valuation equation (23) for the RA problem discounts real returns to capital at the rate $\beta\mu^*_{t+1}/\mu^*_t = \beta\Phi(t;t+1)$, under our $\lambda = \mu^*$ condition. The version of that same equation with unstarred μ of course uses a potentially different discount factor, $\beta\mu_{t+1}/\mu_t = \beta\zeta_{t+1}/\zeta_t = \beta\Theta(t;t+1)$, instead. A sufficient condition for the decentralized version to hold, given that the RA version holds, is that $\mu_{t+1}/\mu_t = \mu^*_{t+1}/\mu^*_t$. In this case it would follow immediately that P_K computed from (23) for the representative agent matches that for the firm in the decentralized model, and it would follow also that, since $\mu = \zeta$, the firm asset valuation equations derived from (20) and (21) are implied by the corresponding consumer equations (12) and (13). In this situation firms, consumers, and representative agent would all be valuing uncertain future returns with the same stochastic discount factor. This is the defining characteristic of a *complete markets* equilibrium. We have verified that all first order conditions for the decentralized model would then be satisfied, and of course the technological constraints (2) and (6) are satisfied in the RA case and therefore also in any corresponding decentralized equilibrium.

All that remains to be verified is that the consumer's budget constraint (7), which occurs slightly rearranged as the firm's budget constraint (9), and transversality and no-Ponzi conditions for the firm and consumer can be satisfied in such a complete markets equilibrium. But it turns out that these conditions do not generally hold when we attempt to verify an equilibrium of the decentralized in which C, K, and L match the solution of the RA model while only a small number of "macro-model" assets (like bonds and equity) are traded.

VII. Why Bonds and equity alone are too few traded assets

We will first make the argument in a general form, then show how it applies to this model. Suppose we have an equation for the evolution of wealth X_t of the form

$$X_t = (1+\rho_t)X_{t-1} - Y_t \,, \tag{33}$$

and we know that

$$E_{t-1}[1+\rho_t] = \beta^{-1} > 1 \,. \tag{34}$$

Then if

$$\beta^s E_t X_{t+s} \underset{s\to\infty}{\to} 0 \,, \tag{35}$$

we can write X_t as

$$X_t = E_t\left[\sum_{s=1}^{\infty} \beta^s Y_{t+s}\right] \,. \tag{36}$$

Now we have two equations determining X_t, the forward solution (36) and the one-period budget constraint (33). These may not be mutually consistent, in which case at least one of the equations we started with—(33), (34), or (35)—must fail to hold.

To check whether the forward solution exists, we can form the innovation in X_t, defined as $\tilde{X}_t = X_t - E_{t-1}X_t$, from both (36) and (33) and see if the expressions can match. That is, we check

$$\tilde{X}_t = \sum_{s=1}^{\infty} \beta^s [E_t - E_{t-1}] Y_{t+s} = \tilde{\rho}_t X_{t-1} - \tilde{Y}_t \,, \tag{37}$$

or, rearranging

$$\tilde{\rho}_t X_t = [E_t - E_{t-1}] \sum_{t=0}^{\infty} \beta^s Y_{t+s} \,. \tag{38}$$

In words, the innovation in the returns on wealth must match the innovation in the discounted present value of Y.

To apply this check to the current model, notice that we have from (15) and (10) that equality of the stochastic discount factors used by firms, consumers, and representative agents implies

$$\frac{D_C U_{t+1}}{D_C U_t} = \frac{\phi'_{t+1}}{\phi'_t} \,. \tag{39}$$

We have already assumed that $\phi'' < 0$. If we assume also that U is concave, (39) can be solved as

$$\pi_t = h(C_t, L_t, C_0, L_0, \phi_0) \,. \tag{40}$$

This lets us rewrite the decentralized model's budget constraint (7), multiplied through by λ_t/λ_{t-1}, as

$$\lambda_t W_t = \frac{\lambda_t}{\lambda_{t-1}} \left(\theta_{t-1}(1 + r_{t-1}) + (1 - \theta_{t-1}) \frac{Q_t + \delta_t}{Q_{t-1}} \right) W_{t-1} - \lambda_t (C_t - h_t - w_t L_t) \,, \tag{41}$$

where $\theta_t = B_t/W_t$ is the fraction of wealth held as bonds. This equation has just the structure we displayed above as (33), with

$$X_t = \lambda_t W_t \tag{42}$$

$$(1 + \rho_t) = \frac{\lambda_t}{\lambda_{t-1}} \left(\theta_{t-1}(1 + r_{t-1}) + (1 - \theta_{t-1}) \frac{Q_t + \delta_t}{Q_{t-1}} \right) \tag{43}$$

and

$$Y_t = \lambda_t (C_t - h_t - w_t L_t) \,. \tag{44}$$

With the definition given in (43) for $1 + \rho_t$, it is easy to verify that the consumer FOC's with respect to S and B, (12) and (13), imply the condition $E_{t-1}[1 + \rho_t] = \beta^{-1}$ that we used in deriving (38). Thus the condition required for supporting the RA allocation with a decentralized equilibrium is

$$W_{t-1}\lambda_{t-1}[E_t - E_{t-1}]\left[\frac{\lambda_t}{\lambda_{t-1}}\left(\theta_{t-1}(1+r_{t-1}) + (1-\theta_{t-1})\frac{Q_t+\delta_t}{Q_{t-1}}\right)\right]$$
$$= [E_t - E_{t-1}]\left[\sum_{t=0}^{\infty}\beta^s\lambda_{t+s}(C_{t+s} - h_{t+s} - w_{t+s}L_{t+s})\right] . \quad (45)$$

The right-hand side of this expression is entirely determined by the stochastic processes of C, L and the marginal product of labor, which are pinned down by the RA equilibrium. The left-hand side is affected by C_t, L_t, δ_t and the portfolio share θ_{t-1}. If there is non-trivial randomness in the return on equity, there will generally be at most one value of the portfolio share (possibly not between zero and one) that makes the left-hand side and right-hand sides match. If the randomness at each date t is concentrated on two points, so that each random variable can take on only two values, "high" and "low", then so long as the stock return $(Q_t + \delta_t)/Q_{t-1}$ is random, there will be a θ_{t-1} that makes (45) hold. But with more general patterns of random variation in the model, it would be a knife-edge special case for it to be possible to make the equation hold. In other words, usually it will be true that if real allocations follow the RA solution and stochastic discount factors match across firms and consumers, and if only equity and bonds are traded, the budget constraint would generate an explosive time path for consumer wealth.

On the other hand, if allowed to engineer the equity return arbitrarily, we can generally describe an equity asset that satisfies (45), even with $B_t \equiv 0$. Such an asset would be engineered so that surprises in its rate of return exactly match surprises in the discounted present value of the excess of consumption over income along the RA solution path.

VIII. A Symmetric Two-Agent Model

Though decentralization into firms and individuals is central to most of macroeconomics, the resulting model is complicated enough that it is worthwhile to look also at a simpler model where these principles can perhaps be brought out more clearly. The model can be interpreted as a rough approximation to a two-country model with international capital flows in a limited menu of assets.

We have two agents, $i = 1, 2$, each of whom receives an endowment $Y_i(t)$ each period. $Y_i(t)$ is i.i.d. across time t and across agents i. There is no possibility of storing the endowment from one period to the next. There are two traded assets, a stock and a bond. Agent i's holdings of stock at t are $S_i(t)$ and of bonds $B_i(t)$. Agent i's objective is to

$$\max_{C,S} E\left[\sum_{t=0}^{\infty}\beta^t U(C_i(t))\right] \quad (46)$$

with constraint

$$C_i(t) + Q_i(t)S_i(t) = Y_i(t) + (Q_t + \delta_t)S_i(t-1) \,. \tag{47}$$

The market-clearing condition is

$$S_1(t) + S_2(t) = 0 \,, \tag{48}$$

so assets acquired by one agent are issued by the other—S is in zero net supply. Adding the two budget constraints and using the market clearing condition gives us the social resource constraint,

$$C_1(t) + C_2(t) = Y_1(t) + Y_2(t) \,. \tag{49}$$

A social planner who put equal weight on the welfare of the two agents would solve

$$\max_{C_1, C_2} E\left[\sum_{t=0}^{\infty} \beta^t \big(U(C_1(t)) + U(C_2(t))\big)\right] \tag{50}$$

subject to (49). The social planner's problem has no dynamics, because there is no real possibility of trading off consumption at different dates, so the solution is trivial. The FOC's imply

$$U'(C_1(t)) = U'(C_2(t)) \,, \text{ all } t \,, \tag{51}$$

which in turn implies (assuming U is differentiable and strictly concave)

$$C_1(t) = C_2(t) = \frac{Y_1(t) + Y_2(t)}{2} = \bar{Y}_t \,. \tag{52}$$

The Euler equations for agent i in the decentralized model are

$$\partial C: \qquad U'(C_i(t)) = \lambda_i(t) \tag{53}$$

$$\partial S: \qquad Q_t \lambda_i(t) = \beta E_t[\lambda_i(t+1)(Q_{t+1} + \delta_{t+1})] \,. \tag{54}$$

Transversality for agent i is

$$\limsup_{t \to \infty} \beta^t E[-\lambda_i(t) Q_t dS_t] \leq 0 \,. \tag{55}$$

As usual we will assume that there is a no-Ponzi constraint bounding the rate of decrease of QS and thereby the rate at which $Q_t dS_t$ can explode toward negative infinity. This allows us to treat the lim sup in (55) as an ordinary lim and to replace the inequality with an equality.

We can now ask whether the planner's allocation, in which available resources are simply split equally between the two agents each period, can be supported as a competitive equilibrium with trading in the single asset S. The agents' budget constraint (47) can be rearranged and multiplied by $\lambda_i(t)$ to be in the form of our abstract budget constraint (33):

$$\lambda_i(t) Q_t S_i(t) = \frac{\lambda_i(t)(Q_t + \delta_t)}{\lambda_i(t-1) Q_{t-1}} \lambda_i(t-1) Q_{t-1} S_i(t-1) - \lambda_i(t)(C_i(t) - Y_i(t)) \,, \tag{56}$$

in which λQS plays the role of X, $\lambda_i(t)(Q_t+\delta_t)/(\lambda_i(t)Q_{t-1})$ the role of $(1+\rho_t)$, and $\lambda_i(t)(C_i(t)-Y_i(t))$ the role of Y_t. The Euler equations guarantee that the condition

$$E_{t-1}[1+\rho_t] = E_{t-1}\left[\frac{\lambda_i(t)(Q_t+\delta_t)}{\lambda_i(t-1)Q_{t-1}}\right] = \beta^{-1} \tag{57}$$

is satisfied. The condition for existence of an equilibrium satisfying transversality and no-Ponzi constraints while implementing the planner's allocation is then that the innovation in

$$\lambda_i(t-1)Q_{t-1}S_i(t-1)\cdot\frac{\lambda_i(t)(Q_t+\delta_t)}{\lambda_i(t-1)Q_{t-1}} = S_i(t-1)\lambda_i(t)(Q_t+\delta_t) \tag{58}$$

match that in

$$\sum_{s=0}^{\infty}\beta^s\lambda_i(t+1)(C_i(t+s)-Y_i(t)) \,. \tag{59}$$

Since $C_i(t)=\bar{Y}_t=\frac{1}{2}(Y_1(t)+Y_2(t))$ for all t, and since $Y_i(t)$ is i.i.d. across both t and i, the expectation of (59) based on information at $t-1$ is zero and its innovation is just

$$U'(\bar{Y}_t)\cdot\left(\tfrac{1}{2}(Y_j(t)-Y_i(t))\right) , \tag{60}$$

where $j\neq i$.[4] Thus to implement the equilibrium, we have to be sure that the unpredictable component of the asset's return moves precisely as (60).

Obviously if the asset is a bond, so that $Q\equiv 1$ and $\delta_t=r_{t-1}$, there is no surprise element at all in the asset's yield and there is therefore no possibility of implementing the planner's allocation with bond trading—except in the special case where there is no uncertainty about the Y's, so they are constant.[5]

To see that it is possible to support the planner's allocation with a single asset, consider the case where $\delta_t=Y_1(t)-Y_2(t)$.[6] To find the return on this asset we need first to determine how its price Q_t behaves. Solving the S Euler equation (54) forward gives us

$$Q_t\lambda_t = E_t\left[\sum_{t=1}^{\infty}\beta^s\lambda_i(t+s)\delta_{t+s}\right] \,. \tag{61}$$

[4]To get this conclusion we use the fact that if Y_1 and Y_2 are two identically distributed, independent random variables, $E[f(Y_1+Y_2)\cdot(Y_1-Y_2)]=0$, regardless of the form of f, because the product is symmetrically distributed about zero by construction.

[5]If the Y's are non-constant, but move in perfectly predictable patterns, then it is possible to implement the planner's solution with bonds. This case does not fit the framework of these notes, because we have assumed i.i.d. Y's, but will be taken up by George Hall in this week's lectures.

[6]Since return on the asset is sometimes negative, this asset is like a partner's share of a firm, where losses of the firm become negative returns to the share owner, rather than like equity shares in a limited liability company.

Since in the planner's equilibrium $\lambda_i(t) = U'(\bar{Y}_t)$, and the Y's are i.i.d., the right-hand side of this equation is just a constant, so

$$Q_t U'(\bar{Y}_t) = \frac{E\left[\big(Y_1(t) - Y_2(t)\big)U'(\bar{Y}_t)\right]}{\beta^{-1} - 1} = 0\,. \tag{62}$$

In other words, so long as the model does not allow satiation ($U' > 0$), this asset trades at a constant price of zero.

With the asset free, the budget constraint (47) simplifies to

$$C_i(t) = Y_i(t) + S_i(t-1) \cdot (Y_1(t) - Y_2(t))\,. \tag{63}$$

To implement the RA allocation, then, agent one must be choosing $S_1(t) \equiv -\frac{1}{2}$. That this does indeed correspond to optimal behavior for both agents is guaranteed by our having determined Q from the agents' FOC's.

What if the single asset paid a dividend equal, say, to $\delta_t = Y_1(t)$? The dividend yield of this asset of course has no chance of matching the innovation defined in (59) under the RA allocation, but Q will generate random variation in the yield through capital gains. One might hope that this would allow the RA allocation to be achieved. Applying (61) gives us

$$Q_t U'(\bar{Y}_t) = \frac{E[U'(\bar{Y}_t)Y_1(t)]}{\beta^{-1} - 1}\,. \tag{64}$$

The right-hand side of this equation is a constant, so Q_t simply moves in inverse proportion to $U'(\bar{Y}_t)$. The condition for existence, matching the innovation in (58) to (60), then reduces to the requirement that the innovations in $S_1(t-1)U'_t Y_1(t)$ and in $U'_t \cdot \big(Y_1(t) - Y_2(t)\big)$ match. Since taking the innovation is a linear operation, this can occur only if $U'_t \cdot \big((1 - S_1(t-1))Y_1(t) - Y_2(t)\big)$ is non-stochastic, which can't be true, no matter what $S_1(t-1)$ is, if the $Y_i(t)$'s are non-deterministic and i.i.d.

IX. Incomplete Markets Solutions for the Two-Country Model

So far we have only checked whether a competitive decentralized equilibrium can support the planner's allocation. If not, what does the equilibrium look like?

In the case where only a bond trades, it may easily be that the only equilibrium is one with no trade in assets at all. Suppose U is unbounded above, $U(0) = 0$, C is constrained to be non-negative, and the distribution of $Y_i(t)$ is such that for every $\varepsilon > 0$, $P[Y_i(t) < \varepsilon] > 0$. We know that $C_i(t) < Y_1(t) + Y_2(t)$, all t, so that in any equilibrium

$$E_t\left[\sum_{t=0}^{\infty} \beta^t U(C_i(t))\right] \leq U(Y_1(t) + Y_2(t)) + \frac{E[Y_1(t) + Y_2(t)]}{\beta^{-1} - 1}\,. \tag{65}$$

Thus if ever $B_i(t)$ grows so large that $U(B_i(t))$ exceeds the right-hand side of (65), agent i would see the possibility, by selling his holdings of B and consuming the proceeds, of increasing his utility beyond any feasible level. Therefore no equilibrium can produce a non-zero probability of B exceeding this level. But as soon as either agent has a negative B (i.e. has done some borrowing), the probability of B becoming arbitrarily large has become non-zero. This is true because there is in each period a non-zero probability that Y_i is less than, say, half the interest due on i's debt. If this happens over and over again, for long enough, B must grow arbitrarily large. Of course this may be extremely unlikely, but so long as $Y_i(t)$ has a positive probability each period of being this small, the probability of it staying this small long enough to push B above the critical level is non-zero. Thus neither agent can issue a bond that he can promise with probability one to service. There is always some chance that after the first debt issue, debt will be forced to grow to a point where lenders will not provide further loans and interest cannot be paid. The model is therefore one in which the borrowing constraint faced by each agent is always binding.

Nonetheless, it is not uncommon to linearize a model like this one around its steady state, ignoring borrowing constraints, and solve it for the bonds-only case. Such a linearized solution will be different from the linearization of the planner's solution, but even more different from the linearization of the true bonds-only equilibrium in which no bonds are in fact issued. This discrepancy illustrates a general point: linearizations of models like these that ignore borrowing constraints are good approximations only over periods in which the borrowing constraints are not binding. The interpretation of a linearized bonds-only equilibrium must be that the bonds are not truly unconditional promises to repay without uncertainty, but that they include a bankruptcy provision that is invoked only in circumstances that are unlikely and remote in time from the present. Then the linearized bonds-only equilibrium is likely to be a good approximation over periods during which invocation of the bankruptcy provision remains a remote possibility.

X. Exercises

1. Consider a consumer whose budget constraint and objective function are (47) and (46) as in the "two-country" model of these notes, with $i = 1$. Suppose the traded asset in the economy has dividend $\delta_t = Y_2(t)$, with Y_2 i.i.d., with the same distribution as Y_1, this consumer's endowment, but with the Y_1 and Y_2 processes independent. Suppose this consumer has $U(C_1(t)) = \log(C_1(t))$ and chooses $S_1(t) = 1$ for all t. Find the implied price process $Q(t)$ for the asset, as a function of constants (which will depend on the expectation of one or more functions of Y_1 and Y_2) and of Y_1 and Y_2. Note that part of the task is to rid the formula of the endogenous variable C_1.
2. Linearize the two-country model about its deterministic steady state with $S = 0$. (The steady state is not unique. There is one for each value of S, because S drifts in equilibrium.) Assume the utility function is $\log(C)$. Treat Y_1, Y_2 and δ as distinct exogenous disturbances. Assume Y_1, Y_2 , and δ each have a mean of 1 and that $\beta = .9$ when computing the deterministic steady state. Solve the system to show how the $C_i(t)$'s and Q_t are determined from past data and current shocks

to the Y's and δ. You will want to suppress roots of β^{-1} are larger, but allow roots equal to one in absolute value.

Compare your results with the linearized version of the planner's solution (which here is trivial—$C_i(t) \equiv \bar{Y}_t$ is both the exact and the linearized solution of the planner's problem.)

Note that it is probably best to use the social resource constraint (49) and one of the two budget constraints (47), which allows dropping the market clearing constraint $S_1 + S_2 = 0$ and using just one S variable. You should, after incorporating Euler equations for the two agents and eliminating Lagrange multipliers, end up with four equations in the four endogenous variables C_1, C_2, Q and S. The resulting linear equation system can be handled analytically, because its Γ_0 matrix can be arranged into a block triangular system with two two-by-two blocks. However, it is probably not worth the effort to solve it analytically, unless perhaps you are using Mathematica, which can do computer algebra. Convert the matrices to numbers and do the eigenvector and eigenvalue calculations with Gauss or Matlab instead.